M. Bordag*

Universität Leipzig, Institut für Theoretische Physik
Augustusplatz 10, 04109 Leipzig, Germany

V.V. Nesterenko†, I.G. Pirozhenko‡

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna, 141980, Russia

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The high temperature asymptotics of the Helmholtz free energy of electromagnetic field subjected to boundary conditions with spherical and cylindrical symmetries are constructed by making use of a general expansion in terms of heat kernel coefficients and the related determinant. For this, some new heat kernel coefficients and determinants had to be calculated for the boundary conditions under consideration. The obtained results reproduce all the asymptotics derived by other methods in the problems at hand and involve a few new terms in the high temperature expansions. An obvious merit of this approach is its universality and applicability to any boundary value problem correctly formulated.

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I. INTRODUCTION

The Casimir effect is one of the interesting phenomena in quantum field theory. Since its discovery more than 50 years ago it attracted much attention. In the past years the interest intensified after its experimental verification reached the one percent level of precision [1–3].

The influence of temperature on the Casimir effect was an important topic since its first experimental demonstration [4] which had been done at room temperature. It was first shown in [5] that the temperature influence was just below what had been measured. It is expected that the temperature contributions will be seen in the upcoming series of experiments.

In quantum field theory, finite temperature can be described at equilibrium in the Matsubara formalism by imposing periodic (resp. antiperiodic for fermions) boundary conditions in the imaginary time coordinate. Technically this is very similar to the Casimir effect for plane boundaries and can mathematically be described by the same Riemann and Hurwitz zeta functions. Another formula important for applications is the well known Lifshitz formula describing the interaction between plane dielectric bodies at finite temperature. In the case of non flat boundaries the situation is, however, more complicated. In oder to obtain the Casimir effect at finite temperature one has to know it at least at zero temperature, i.e., one has to know the spectrum of the corresponding operator. Even then usually complicated calculations are necessary and explicit results are rare.

An opposite situation occurs with the asymptotic expansion of the Casimir energy at high temperature. It turns out to be determined to a large extent by local quantities which are much easier to obtain. In the paper [6], which did not receive the due attention, it was shown that this expansion can be written down in terms of the heat kernel coefficients and the functional determinant of the operator corresponding to the spatial part of the problem. As known, the heat kernel coefficients for differential operators on manifolds with and without boundary are known to depend on the properties of this manifold only locally. This means that they can be represented as integrals over the manifold and over its boundary [7] whereby the characteristics like curvature enter as local functions. The calculation of these coefficients is a topic of its own and much progress had been achieved especially during the past decade, see the book [8] for example. Less is known about the determinant. It is not known in general whether it is a local quantity. However, from several examples it is known to be calculable much easier than the corresponding Casimir energy at

*E-mail: Michael.Bordag@itp.uni-leipzig.de
†E-mail: nestr@thsun1.jinr.ru
‡E-mail: pirozhen@thsun1.jinr.ru

zero temperature. As a consequence, the high temperature expansion of the Casimir energy can be calculated quite easily. This was emphasised in the recent review [9] where the basic formula (2.10) is taken from.

In the present paper the we apply these general formulas to several specific examples. First we consider parallel plates in order to demonstrate the technical tools on a simple example. Then we consider the conducting spherical and cylindrical shells obtaining new terms in the asymptotic expansion. Eventually we consider the dielectric ball and cylinder where we restrict ourselves to the dilute approximation. These examples demonstrate the effectiveness of the method.

An interesting application of the general formula is the discussion of the so called classical limit which had recently be discussed in [21]. It is understood as to take place if the internal energy which is connected with the free energy by Eq. (2.11) (see below the next section) tends to zero for $T \to \infty$. This happens if the heat kernel coefficients with number $n \leq \frac{3}{2}$ vanish.

The first calculation of the leading contributions to the high temperature asymptotics of the Casimir energy for curved boundaries was given in [10] using the multiple reflection expansion. As it was noticed in the recent paper [11] the multiple reflection expansion can be used for the calculation of the heat kernel coefficients demonstrating the equivalence of both approaches up to the question of the determinant.

The situation is to some extend different for boundaries with edges and corners. Here the application of Riemann and Hurwitz zeta functions seems to be more appropriate. A first example of this kind was given in [12]. The appropriate more general methods can be expected to be those given in [13].

The layout of the paper is as follows. In Sec. II the derivation of the high temperature expansions in terms of the heat kernel coefficients is briefly given. In Sec. III the original setting of the Casimir effect, i.e. parallel perfectly conducting plates in vacuum, is considered and the high temperature asymptotics of the thermodynamic functions are derived in terms of the relevant heat kernel coefficients. In Sec. IV the high temperature asymptotics for electromagnetic field with boundary conditions on a sphere are obtained. In Sec. V the high temperature expansions are constructed for the boundary conditions defined on the lateral of a circular infinite cylinder. The heat kernel coefficients needed are calculated by making use of the respective zeta functions that have been obtained in an explicit form in terms of the Riemann zeta function in Ref. [14] and also by applying the results of Ref. [15]. The functional determinants entering the asymptotic expansions at hand are calculated by making use of the technique developed in Ref. [17]. The results obtained are compared with the high temperature asymptotics which have been derived for boundary conditions under consideration by other methods. The possible extension of the approach is discussed in the Conclusions (Sec. VI).

The mathematical details of the calculation of the zeta determinants are presented in Appendix A for electromagnetic field subjected to boundary conditions given on a sphere and in Appendix B for the boundary conditions defined on the lateral of an infinite circular cylinder.

II. HEAT KERNEL COEFFICIENTS AND HIGH TEMPERATURE EXPANSIONS

Let the dynamics of quantum field be defined by the operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{2.1}$$

where Δ is not of necessity the Laplace operator, but an elliptic differential operator depending only on space coordinates. The free energy F of the field is determined by the zeta function $\zeta_{\mathrm{T}}(s)$ corresponding to the Euclidean version of the operator (2.1)

$$F = -\frac{T}{2}\,\zeta'_{\mathrm{T}}(0). \tag{2.2}$$

Here T is the temperature measured in energy units (the Boltzmann constant k_{B} is assumed to be equal to 1), and the zeta function $\zeta_{\mathrm{T}}(s)$ is defined in a standard way

$$\zeta_{\mathrm{T}}(s) = \sum_{m=-\infty}^{\infty}\sum_{\{k\}}\left(\Omega_m^2 + \omega_k^2\right)^{-s}, \tag{2.3}$$

with $\Omega_m = 2\pi m T/\hbar$ being the Matsubara frequencies and ω_k^2/c^2 standing for the eigenvalues of the operator $-\Delta$ in Eq. (2.1)

$$-\Delta\varphi_k(\boldsymbol{x}) = \frac{\omega^2}{c^2}\varphi_k(\boldsymbol{x}). \tag{2.4}$$

The characteristics of the quantum field system with dynamical operator (2.1) at zero temperature are determined by the zeta function $\zeta(s)$ associated with the operator $-\Delta$

$$\zeta(s) = \sum_{\{k\}} \omega_k^{-2s}. \tag{2.5}$$

From the mathematical point of view the zeta function $\zeta(s)$ corresponding to the space part of the operator (2.1) is, undoubtedly, a simpler object than the complete zeta function $\zeta_{\rm T}(s)$ because the definition (2.3) involves an additional sum over the Matsubara frequencies. Here a natural question arises whether one can gain knowledge of the quantum field at nonzero temperature possessing only the zeta function $\zeta(s)$. In Ref. [6] it was shown that proceeding from the zeta function $\zeta(s)$ one can deduce the high temperature asymptotics of the thermodynamic functions such as Helmholtz free energy, internal energy, and entropy. Let us remind briefly the derivation of these asymptotics. By making use of the formula

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, e^{-\lambda t} \tag{2.6}$$

the zeta function (2.3) can be represented in the form

$$\zeta_{\rm T}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \sum_{m=-\infty}^{\infty} e^{-\Omega_m^2 t} \sum_{\{k\}} e^{-\omega_k^2 t}. \tag{2.7}$$

The term with $m = 0$ in this formula gives the zeta function (2.5). In the remaining terms we substitute the heat kernel $K(t)$ of the operator $-\Delta$ by its asymptotic expansion at small t

$$K(t) \equiv \sum_{\{k\}} e^{-\omega_k^2\, t} \simeq \frac{1}{(4\pi t)^{3/2}} \sum_{n=0,1/2,\ldots} a_n\, t^n + \ldots\,. \tag{2.8}$$

As a result we arrive at the following asymptotic representation for the complete zeta function $\zeta_{\rm T}(s)$

$$\zeta_{\rm T}(s) \simeq \zeta(s) + \frac{2}{(2\pi)^{3/2}} \sum_{n=0,1/2,\ldots} a_n \left(\frac{\hbar}{2\pi T}\right)^{2s-3+2n} \frac{\Gamma(s-3/2+n)}{\Gamma(s)}\, \zeta_{\rm R}(2s+2n-3), \tag{2.9}$$

where $\zeta_{\rm R}(s)$ is the Riemann zeta function. Taking the derivative of the right hand side of Eq. (2.9) at the point $s = 0$ and substituting the result into Eq. (2.2) one obtains the high temperature expansion for the free energy

$$\begin{aligned} F(T) \simeq {}& -\frac{T}{2}\zeta'(0) + a_0 \frac{T^4}{\hbar^3}\frac{\pi^2}{90} - a_{1/2}\frac{T^3}{4\pi^{3/2}\hbar^2}\zeta_{\rm R}(3) - \frac{a_1}{24}\frac{T^2}{\hbar} + \frac{a_{3/2}}{(4\pi)^{3/2}}\, T\, \ln\frac{\hbar}{T} \\ & -\frac{a_2}{16\pi^2}\,\hbar \left[\ln\left(\frac{\hbar}{4\pi T}\right) + \gamma\right] - \frac{a_{5/2}}{(4\pi)^{3/2}}\frac{\hbar^2}{24 T} \\ & -T\sum_{n\geq 3} \frac{a_n}{(4\pi)^{3/2}} \left(\frac{\hbar}{2\pi T}\right)^{2n-3} \Gamma(n-3/2)\, \zeta_{\rm R}(2n-3)\,. \end{aligned} \tag{2.10}$$

Here γ is the Euler constant. The argument of the logarithms in expansion (2.10) are dimensional, but upon collecting similar terms with account for the logarithmic ones in $\zeta'(0)$ it is easy to see that finally the logarithm function has a dimensionless argument, at least for $a_2 = 0$. Let us note that according to the definition (2.8) the heat kernel coefficients in our consideration are dimensional, because the frequencies ω_k have the dimensionality $[\text{time}]^{-1}$.

The asymptotic expansions for the internal energy $U(T)$ and the entropy $S(T)$ are deduced from Eq. (2.10) employing the thermodynamic relations

$$U(T) = -T^2\frac{\partial}{\partial T}\left(T^{-1}F(T)\right), \tag{2.11}$$

$$S(T) = T^{-1}\left(U(T) - F(T)\right) = -\frac{\partial F}{\partial T}. \tag{2.12}$$

Substituting the expansion (2.10) into Eqs. (2.11) and (2.12) one arrives at the asymptotics

$$
\begin{aligned}
U(T) \simeq\, & a_0 \frac{T^4}{\hbar^3}\frac{\pi^2}{30} + a_{1/2}\frac{T^3}{\hbar^2}\frac{\zeta_{\rm R}(3)}{2\,\pi^{3/2}} + a_1\frac{T^2}{24\,\hbar} + \frac{a_{3/2}}{(4\pi)^{3/2}}T \\
& - a_2\frac{\hbar}{16\pi^2}\left[\ln\left(\frac{\hbar}{4\pi T}\right) + \gamma + 1\right] - \frac{a_{5/2}}{(4\pi)^{3/2}}\frac{\hbar^2}{12\,T} \\
& - \frac{T}{4\,\pi^{3/2}}\sum_{n\geq 3} a_n \left(\frac{\hbar}{2\pi T}\right)^{2n-3}\Gamma(n-1/2)\,\zeta_{\rm R}(2n-3),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
S(T) \simeq\, & \frac{1}{2}\zeta'(0) + a_0\frac{T^3}{\hbar^3}\frac{2\,\pi^2}{45} + a_{1/2}\frac{T^2}{\hbar^2}\frac{3}{4}\frac{\zeta_{\rm R}(3)}{\pi^{3/2}} + a_1\frac{T}{12\,\hbar} \\
& + \frac{a_{3/2}}{(4\pi)^{3/2}}\left(1 - \ln\frac{\hbar}{T}\right) - a_2\frac{\hbar}{16\pi^2 T} - \frac{a_{5/2}}{(4\pi)^{3/2}}\frac{\hbar^2}{24\,T^2} \\
& - \frac{1}{4\,\pi^{3/2}}\sum_{n\geq 3} a_n\left(\frac{\hbar}{2\pi T}\right)^{2n-3}(n-2)\,\Gamma(n-3/2)\,\zeta_{\rm R}(2n-3)\,.
\end{aligned} \tag{2.14}
$$

In Eq. (2.13) the term proportional to a_2 contains the logarithm of dimensional quantity: $[\hbar/T] = [\text{time}]^{-1}$. This is the result of the arbitrariness coming in from the ultraviolet divergences in case of $a_2 \neq 0$, see [26] for a more detailled discussion. Unlike this situation, collecting the logarithm functions in the $a_{3/2}$-term and in $\zeta'(0)$ in Eq. (2.14) leads to a dimensionless argument of the logarithm in the final expression.

It is worth noting that the zeta determinant of the operator $-\Delta$ (i. e. $\zeta'(0)$) does not enter the asymptotic expansion for the internal energy (2.13). Therefore this high temperature expansion is completely defined only by the heat kernel coefficients. In view of this, the first term in the asymptotics of the free energy in Eq. (2.10) is referred to as a pure entropic contribution. Its physical origin is till now not elucidated.

III. PERFECTLY CONDUCTING PARALLEL PLATES IN VACUUM

In this section we demonstrate the application of the high temperature expansions (2.10), (2.13), and (2.14) to a simple problem of electromagnetic field confined between two perfectly conducting parallel plates in vacuum. First, we briefly recall how to construct the zeta function in this problem.

As well known, for example, from the theory of waveguides and resonators [18] the vectors of electric and magnetic fields in the problem at hand are expressed in terms of the electric ($\mathbf{\Pi}'$) and magnetic ($\mathbf{\Pi}''$) Hertz vectors, each having only one nonzero component $\Pi_z^{'}$ and $\Pi_z^{''}$ satisfying, respectively, Dirichlet and Neumann conditions on the internal surface of the plates. The functions $\Pi_z^{'}$ and $\Pi_z^{''}$ obey the equations

$$
\left(\frac{\partial^2}{\partial z^2} + \boldsymbol{\nabla}^2\right)\Pi_z^{'} = \frac{\omega^2}{c^2}\Pi_z^{'}, \quad \left(\frac{\partial^2}{\partial z^2} + \boldsymbol{\nabla}^2\right)\Pi_z^{''} = \frac{\omega^2}{c^2}\Pi_z^{''}, \tag{3.1}
$$

where ω is the frequency of electromagnetic oscillations, $\boldsymbol{\nabla}^2$ stands for the two-dimensional Laplace operator for the variables $(x, y) = \mathbf{x}$. The separation of variables results in the following solution,

$$
\begin{aligned}
\Pi_z^{'}(\mathbf{x}, z) &= \exp(i\mathbf{kx})\,\cos\left(\frac{n\pi}{a}\right), \quad n = 0,\, 1,\, 2, \ldots, \\
\Pi_z^{''}(\mathbf{x}, z) &= \exp(i\mathbf{kx})\,\sin\left(\frac{n\pi}{a}\right), \quad n = 1,\, 2, \ldots, \\
\omega_n^2(\mathbf{k}) &= c^2\left[\mathbf{k}^2 + \left(\frac{n\pi}{a}\right)^2\right],
\end{aligned} \tag{3.2}
$$

where a is the distance between the plates. Hence, the states of electromagnetic field with the energy $\hbar\omega_n$, $n \geq 1$, are doubly degenerate, while the state with the energy $\hbar\omega_0 = \hbar c k$ is nondegenerate.

With allowance for this the zeta function in the problem under consideration is given by

$$
\zeta(s) = \frac{L_x\, L_y}{c^{2s}}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\left\{2\sum_{n=1}^{\infty}\left[\mathbf{k}^2 + \left(\frac{n\pi}{a}\right)^2\right]^{-s} + \left(\mathbf{k}^2 + \mu^2\right)^{-s}\right\}, \tag{3.3}
$$

where L_x and L_y are the dimensions of the plates.

For a correct definition of the integral in this formula in the small $\mathbf{k}$ region the photon mass μ is introduced (infrared regularization). At the final step of calculations one should put $\mu = 0$. On integrating in Eq. (3.3) and substituting the sum over n by the Riemann zeta function one arrives at the result

$$\zeta(s) = \frac{L_x\,L_y}{2\pi\,c^{2s}}\left[\left(\frac{\pi}{a}\right)^{2-2s}\frac{\zeta_{\rm R}(2s-2)}{s-1}+\frac{1}{2}\frac{\mu^{2-2s}}{s-1}\right]. \tag{3.4}$$

The zeta function (3.4) gives the well-known value for the Casimir energy

$$E_{\rm C} = \frac{\hbar}{2}\,\zeta\left(-\frac{1}{2}\right) = -c\,\hbar\,\frac{\pi^2}{720}\,\frac{L_xL_y}{a^3} \tag{3.5}$$

or for its density

$$\frac{E_{\rm C}}{V} = -\frac{c\hbar\pi^2}{720a^4}, \quad \text{where} \quad V = a\,L_x\,L_y. \tag{3.6}$$

In order to construct the high temperature expansions (2.10), (2.13), and (2.14) the heat kernel coefficients for the system under consideration should be obtained by making use of the zeta function (3.4).

The zeta function (3.4) or, in the general case, (2.5) and the corresponding heat kernel (2.8) are related via the Mellin transform

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\,K(t). \tag{3.7}$$

This enables one to express the heat kernel coefficients a_n in terms of the values of the zeta function at the respective points

$$\frac{a_n}{(4\pi)^{3/2}} = \lim_{s\to\frac{3}{2}-n}(s+n+3/2)\,\zeta(s)\,\Gamma(s), \qquad n = 0,\,1/2,\,\dots\,. \tag{3.8}$$

Substituting Eq. (3.4) into Eq. (3.8) we obtain for perfectly conducting parallel plates only one nonzero coefficient a_0

$$a_0 = 2\,\frac{V}{c^2}, \tag{3.9}$$

where $V = L_x\,L_y\,a$ is the volume of the space bounded by the plates. This is just an illustration of the well-known fact that for flat manifolds without boundary or with flat boundary all the heat kernel coefficients except for a_0 vanish [19]. It should be noted here that we are considering only electromagnetic field confined between the plates and do not take into account that field outside the plates.

From Eqs. (2.13) and (3.9) it follows that the density of internal energy has the following high temperature asymptotics

$$\frac{U(T)}{V} \simeq 4\,\frac{\sigma}{c}\,T^4, \quad T\to\infty, \tag{3.10}$$

where σ is the Stefan-Boltzmann constant

$$\sigma = \frac{\pi^2\,k_{\rm B}^4}{60\,c^2\,\hbar^3}. \tag{3.11}$$

Recall that in our formulae we put $k_{\rm B} = 1$, that is, the temperature is measured in energy units. The transition to degrees is performed by the substitution $T \to k_{\rm B}\,T$.

When calculating the high temperature asymptotics of the free energy (2.10) and the entropy (2.14) one needs to derive $\zeta'(0)$ for the zeta function (3.4). Keeping in mind that $\zeta_{\rm R}(-2) = 0$ it is convenient to use here the Riemann reflection formula

$$2^{1-s}\,\Gamma(s)\,\zeta_{\rm R}(s)\,\cos(\pi\,s/2) = \pi^2\zeta_{\rm R}(1-s) \tag{3.12}$$

which yields

$$\zeta_{\rm R}(2s-2) \underset{s\to 0}{\simeq} -s\,\frac{\zeta_{\rm R}(3)}{2\pi^2}+\mathcal{O}(s^2). \tag{3.13}$$

From here we deduce

$$\zeta'(0)=\frac{L_x\,L_y}{4\,\pi\,a^2}\,\zeta_{\rm R}(3)=\frac{V}{4\,\pi\,a^3}\,\zeta_{\rm R}(3). \tag{3.14}$$

Insertion of Eqs. (3.9) and (3.14) into Eq. (2.10) gives the following high temperature behaviour for the density of free energy

$$\frac{F}{V}\simeq-\frac{T}{8\,\pi\,a^3}\,\zeta_{\rm R}(3)-\frac{T^4}{c^3\,\hbar^3}\,\frac{\pi^2}{90}. \tag{3.15}$$

As was noted above, we are considering only electromagnetic field between the plates. Therefore when calculating the Casimir forces one should drop the last term in Eq. (3.15) since its contribution is canceled by the pressure of the black body radiation on the outward surfaces of the plates. As a result the high temperature asymptotics of the Casimir force, per unit surface area, attracting two perfectly conducting plates in vacuum is

$$\mathcal{F}\simeq-\frac{T}{4\pi a^3}\zeta_{\rm R}(3). \tag{3.16}$$

Usually in the Casimir calculations the contribution of the free black body radiation is subtracted from the very beginning [20].

It is interesting to note that the Casimir force (3.16) and the first term on the right hand side of Eq. (3.15) are pure classical quantities because they do not involve the Planck constant $\hbar$. These classical asymptotics seem to be derivable without appealing to the notion of quantized electromagnetic field. The classical limit of the theory of the Casimir effect is discussed in a recent paper [21].

Employing Eqs. (2.12) and (3.15) one arrives at the high temperature behavior of the entropy density

$$\frac{S(T)}{V}\simeq\frac{\zeta_{\rm R}(3)}{8\pi a^3}+\frac{2\,T^3\,\pi^2}{45\,c^3\,\hbar^3}. \tag{3.17}$$

It is worth noticing that the corrections to Eqs. (3.10), (3.15), (3.17) are exponentially small.

The example considered shows that the zeta function of the spatial part of evolution operator really enables one to obtain the high temperature asymptotics of the thermodynamic functions in a straightforward way. In the ensuing sections we shall consider quantum fields defined on manifolds with boundaries possessing spherical or cylindrical symmetries, when the relevant zeta functions cannot be obtained in a closed form. Furthermore in these cases the spectrum of the operator $-\Delta$ is not known explicitly. And nevertheless the method proposed is applicable to these cases also.

IV. THERMODYNAMIC ASYMPTOTICS FOR ELECTROMAGNETIC FIELD WITH BOUNDARY CONDITIONS ON A SPHERE

In the present section we consider electromagnetic field subjected to three types of boundary conditions on the surface of a sphere: i) an infinitely thin and perfectly conducting spherical shell; ii) the surface of a sphere delimits two material media with the same velocity of light; iii) a dielectric ball placed in unbounded dielectric medium. In order to obtain the heat kernel coefficients determining the high temperature asymptotics (2.10), (2.13), and (2.14) it is convenient to use the explicit representation of the relevant spectral zeta functions in terms of the Riemann zeta function. These formulae were derived in our recent paper [14] by taking into account the first two terms of the uniform asymptotic expansion for the product of the modified Bessel functions $I_\nu(\nu z)\,K_\nu(\nu z)$.

A. Perfectly conducting spherical shell

We take advantage of Eq. (2.26) in Ref. [14] substituting there the variable s by $2s$ and recovering the explicit dependence on the velocity of light c. The latter results in the replacement of the sphere radius by R/c:

$$\zeta(s) \simeq \frac{1}{4}\left(\frac{R}{c}\right)^{2s} s\,(1+s)\,(2+s)\left\{\left(2^{1+2*s}-1\right)\,\zeta_{\rm R}(1+2s)\;-\;2^{1+2s}\right.$$
$$\left.+\;q(s)\left[\left(2^{3+2s}-1\right)\,\zeta_{\rm R}(3+2s)\;-\;2^{3+2\,s}\right]+\ldots\right\}, \tag{4.1}$$

where

$$q(s)=\frac{1}{3840}\,(480\;+\;1736\;+\;2016\,s^2\;+\;568\,s^3), \tag{4.2}$$

and R is the radius of a sphere. The terms omitted in Eq. (4.1) are of the form

$$q_k(s)\left[\left(2^{2(k+s)+1}-1\right)\,\zeta_{\rm R}(2\,k+2\,s+1)-2^{2\,(k+s)+1}\right],\quad k=2,3,4,\ldots, \tag{4.3}$$

where $q_k(s)$ stand for some polynomials in s.

Analysis of Eqs. (4.1) and (4.2) shows that the zeta function (4.1) for a perfectly conducting spherical shell enables one to find the exact values of the first six heat kernel coefficients, namely:

$$a_0=0,\quad a_{1/2}=0,\quad a_1=0,\quad a_{3/2}=2\,\pi^{3/2},\quad a_2=0,\quad a_{5/2}=\frac{\pi^{3/2}}{20}\,\frac{c^2}{R^2}. \tag{4.4}$$

Taking account of the structure of the omitted terms (4.3) it is easy to see that

$$a_j=0,\qquad j=3,4,5,\ldots\,. \tag{4.5}$$

Having obtained the heat kernel coefficients (4.4) and (4.5) we are in position to construct the high temperature asymptotics of the internal energy of electromagnetic field by making use of Eq. (2.13)

$$U(T)\simeq\frac{T}{4}-\left(\frac{c\,\hbar}{R}\right)^2\frac{1}{1920\,T}+\mathcal{O}(T^{-3}). \tag{4.6}$$

In the paper [16] the corresponding heat kernel coefficients are calculated to a much higher order. Taking these values, more terms to this expansion can be easily added.

In order to write the asymptotic expansions (2.10) and (2.14) the derivative of the zeta function at the point $s=0$ should be calculated. Equation (4.1) gives an approximate value for $\zeta'(0)$

$$\zeta'(0)=\frac{\gamma}{2}+\ln 2+\frac{7}{16}\,\zeta_{\rm R}(3)-\frac{9}{8}+\frac{1}{2}\,\ln\frac{R}{c}=0.38265+\frac{1}{2}\,\ln\frac{R}{c}\,. \tag{4.7}$$

The terms omitted in (4.1) will render precise only the first term in the final form of this expression, while the second term $(1/2)\ln(R/c)$ will not change. The exact value of $\zeta'(0)$ is calculated in Appendix A

$$\zeta'(0)=\left[\frac{1}{2}-\frac{\gamma}{2}+\frac{7}{6}\ln 2+6\,\zeta'_{\rm R}(-1)\right]+\left(-\frac{5}{8}+\frac{1}{2}\,\ln\frac{R}{c}+\ln 2+\frac{\gamma}{2}\right)$$
$$=0.38429+\frac{1}{2}\,\ln\frac{R}{c}. \tag{4.8}$$

It is worth noting that the expression in the parentheses is exactly the value of $\zeta'(0)$ for a compact ball with continuous velocity of light on its surface (see Eq. (4.16) in the next subsection). As a result we have the following high temperature asymptotics of the free energy and the entropy in the problem in question

$$F(T)\simeq-\frac{T}{4}\left(\ln\frac{R\,T}{\hbar c}+0.76858\right)-\left(\frac{\hbar\,c}{R}\right)^2\frac{1}{3840\,T}+\mathcal{O}(T^{-3}), \tag{4.9}$$

$$S(T)\simeq 0.44215+\frac{1}{4}\,\ln\frac{R\,T}{\hbar\,c}-\frac{1}{3840}\left(\frac{\hbar\,c}{R\,T}\right)^2+\mathcal{O}(T^{-4})\,. \tag{4.10}$$

The expression (4.9) exactly reproduces the asymptotics obtained in Ref. [10] by making use of the multiple scattering technique (see Eq. (8.39) in that paper). We have not calculated the coefficient $a_{7/2}$, therefore we do not know the sign of the T^{-3}-correction in (4.9). In Ref. [10] it is noted that this term is negative.

In the case under consideration the high temperature asymptotics of the thermodynamic functions contain the terms independent of the Planck constant $\hbar$ or, in other words, classical contributions (see Eqs. (4.6), (4.9), and (4.10)). This is also true for the high temperature limit of the Casimir force calculated per unit area of a sphere

$$\mathcal{F}(T) \simeq -\frac{1}{4\pi R^2}\frac{\partial F(T)}{\partial R} = \frac{T}{16\pi R^2} - \left(\frac{\hbar c}{R}\right)^2 \frac{1}{4\pi R^3}\,\frac{1}{1920\,T} + \mathcal{O}(T^{-3}). \tag{4.11}$$

The leading classical term in the asymptotics (4.11) describes the Casimir force that seeks to expand the sphere. The quantum correction in this formula stands for the Casimir pressure exerted on the sphere surface.

In Eqs. (4.6), (4.9) and (4.10) the Stefan-Boltzman terms proportional to T^4 are absent because the contribution of the Mikowski space was subtracted from the very beginning in our calculations [22]. As a result we obtain the vanishing heat kernel coefficients a_0 and $a_{1/2}$. Therefore, our results describe only the deviation from the Stefan-Boltzmann law caused by the perfectly conducting sphere.

B. Compact ball with equal velocities of light inside and outside

Let us consider the spherical surface that delimits the media with "relativistic invariant" characteristics i.e., the velocity of light is the same inside and outside the sphere [23]. In this problem there naturally arises a dimensionless parameter [24]

$$\xi^2 = \left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)^2 = \left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^2, \tag{4.12}$$

where ε_1 and ε_2 (μ_1 and μ_2) are permittivities (permeabilities) inside and outside the sphere. As usual we perform the calculation in the first order of the expansion with respect to ξ^2.

In order to derive the zeta function for the boundary conditions at hand one should multiply Eq. (4.1) by ξ^2 and replace there $q(s)$ by the polynomial

$$p(s) = -\frac{1}{2}\left[1 - \frac{9}{2}\,(3+s) + \frac{5}{2}\,(3+s)\,(4+s) - \frac{7}{24}\,(3+s)\,(4+s)\,(5+s)\right]. \tag{4.13}$$

The zeta function, obtained in this way, affords the exact heat kernel coefficients up to a_3

$$\begin{gathered} a_0 = 0, \quad a_{1/2} = 0, \quad a_1 = 0, \quad a_{3/2} = 2\,\pi^{3/2}\,\xi^2, \quad a_2 = 0, \\ \frac{a_{5/2}}{(4\pi)^{3/2}} = \xi^2\frac{c^2}{R^2}\frac{p(-1)}{8} = 0, \quad a_3 = 0. \end{gathered} \tag{4.14}$$

Substitution of these coefficients into Eq. (2.13) gives the following high temperature behavior of the internal energy in the problem under consideration

$$U(T) \simeq \xi^2\,\frac{T}{4} + \mathcal{O}(T^{-3})\,. \tag{4.15}$$

The value of $\zeta'(0)$ is calculated in Appendix A

$$\zeta'(0) = \xi^2\left(-\frac{5}{8} + \frac{1}{2}\,\ln\frac{R}{c} + \ln 2 + \frac{\gamma}{2}\right) = \xi^2\left(0.35676 + \frac{1}{2}\ln\frac{R}{c}\right). \tag{4.16}$$

It is this value that is supplied by Eq. (4.1) after changes specified above and with allowance for that $p(-1) = 0$.

By making use of Eqs. (2.10), (4.14), and (4.16) we deduce the high temperature asymptotics for free energy

$$\begin{aligned} F(T) &= -\xi^2\,\frac{T}{4}\,\left(\gamma + \ln 4 - \frac{5}{4}\right) + \frac{\xi^2}{4}\,T\,\ln\frac{\hbar\,c}{RT} + \mathcal{O}(T^{-3}), \\ &= -\xi^2\,\frac{T}{4}0.71352 + \frac{\xi^2}{4}\,T\,\ln\frac{\hbar\,c}{RT} + \mathcal{O}(T^{-3}). \end{aligned} \tag{4.17}$$

The entropy in the present case has the following high temperature behavior

$$S(T) = \frac{\xi^2}{4}\left(1+\gamma+\ln 4-\frac{5}{4}-\ln\frac{\hbar\, c}{R\,T}\right)+\mathcal{O}(T^{-3}),$$
$$= \frac{\xi^2}{4}\left(1.71352-\ln\frac{\hbar\, c}{R\,T}\right)+\mathcal{O}(T^{-3}). \tag{4.18}$$

The asymptotics (4.15) and (4.17) completely coincide with analogous formulae obtained in Refs. [22,25] by the mode summation method combined with the addition theorem for the Bessel functions.

C. Dielectric ball in unbounded dielectric medium

The zeta function for electromagnetic field in the background of a pure dielectric ball ($\mu_1=\mu_2=1,\ \ \varepsilon_1\neq\varepsilon_2$) has not been calculated in an explicit form. In Ref. [26] the heat kernel coefficients up to a_2 were found. Here we use the results of this paper confining ourselves to the Δn^2-approximation, where $\Delta n = n_1-n_2 = n_1\, n_2\,(c_2-c_1)/c \simeq (c_2-c_1)/c$, n_i and c_i are the refractive index and the velocity of light inside ($i=1$) and outside ($i=2$) the ball, and c is the velocity of light in the vacuum. It is assumed that c_1 and c_2 differ from c slightly, therefore c_2-c_1 and Δn are small quantities. In this approximation we have

$$a_0 = 8\,\pi\,\frac{R^3}{c^3}\,(\Delta n+2\,\Delta n^2),\quad a_{1/2}=-4\,\pi^{3/2}\,\frac{R^2}{c^2}\,\Delta n^2,$$
$$a_1=0,\quad a_{3/2}=\pi^{3/2}\,\Delta n^2,\quad a_2=0. \tag{4.19}$$

The coefficients a_1 and a_2 equal zero only in the Δn^2-approximation considered here. In the general case they contain terms proportional to Δn^k, where $k\geq 3$.

Allowance for one more term in the uniform asymptotic expansion of the modified Bessel functions, as compared with the calculations in Ref. [26], gives the next heat kernel coefficient

$$a_{5/2} = \Delta n^2\,\frac{\pi^{3/2}\,c^2}{R^2}. \tag{4.20}$$

Making use of the technique developed in Ref. [17] one obtains the following expression for the derivative of the zeta function for a pure dielectric ball at the point $s=0$ (see Appendix A)

$$\zeta'(0) = \frac{\Delta n^2}{4}\left(-\frac{7}{8}+\ln\,\frac{R}{c}+\ln 4+\gamma\right). \tag{4.21}$$

Dealing with the asymptotics expansions (2.10), (2.13), and (2.14) we have till now not touched on the problem of renormalization. At the first sight it is unnecessary because these expansions are free of divergencies. However, it is well known that all physical predictions, obtained in the framework of the quantum field theory formalism should be expressed in terms of renormalized quantities. Unfortunately the renormalization procedure in the framework of the approach employed here is not formulated in the general case (see discussion of this problem, for example, in Ref. [6]).

Let us turn to the renormalization of the high temperature asymptotics for a dielectric ball. In the general case the Helmholtz free energy of a material body having the volume V and the surface area S can be written in the form

$$F = V\,f+S\,\sigma+F_{\text{ren}}, \tag{4.22}$$

where f is the free energy of a unit volume, σ denotes the surface tension, and F_{ren} is the free energy of electromagnetic field connected with this body and having the temperature T. Bearing in mind the thermal field of a material heated ball we have to consider only F_{ren}.

From Eq. (4.19) it follows that the coefficient a_0 is proportional to the volume of the ball, therefore the corresponding terms in the asymptotic expansions should be "absorbed" by the renormalization of the free energy density f. The coefficient $a_{1/2}$ in Eq. (4.19) is proportional to the surface area of the ball and this coefficient should be involved in the redefinition of the surface tension σ. With regard to all this we obtain the following high temperature behavior of "observed" free energy $F_{\text{ren}}(T)$ in the problem at hand

$$F_{\text{ren}}(T)\simeq -\frac{\Delta n^2}{8}\,T\,\left(\ln\frac{4\,T\,R}{\hbar\,c}+\gamma-\frac{7}{8}\right)+\frac{\Delta n^2}{192}\,\frac{\hbar^2\,c^2}{R^2\,T}+\mathcal{O}(T^{-2}). \tag{4.23}$$

The renormalized high temperature asymptotics for internal energy and for entropy are derived in the same way

$$U_{\rm ren}(T) \simeq \frac{\Delta n^2}{8}\,T + \frac{\Delta n^2}{96}\,\frac{c^2\,\hbar^2}{R^2\,T} + \mathcal{O}(T^{-2}), \tag{4.24}$$

$$S_{\rm ren}(T) \simeq \frac{\Delta n^2}{8}\left(\frac{1}{8} + \gamma + \ln\,\frac{4\,R\,T}{\hbar\,c}\right) + \frac{\Delta n^2}{112}\,\frac{c^2\,\hbar^2}{R^2\,T^2} + \mathcal{O}(T^{-3})\,. \tag{4.25}$$

It is worth comparing these results with analogous asymptotics obtained by different methods. In Ref. [22] at the beginning of calculations the first term of expansion of internal energy (4.24) was derived. The subsequent integration of the thermodynamic relation (2.11) gave the correct coefficient of the logarithmic term in the asymptotics of free energy (4.23). Barton [27] managed to deduce the asymptotics (4.23) – (4.25) except for the contributions proportional to negative powers of T. One should keep in mind that our parameter Δn corresponds to $2\pi\alpha\,n$ in the notations of Ref. [27].

The asymptotics (4.23)–(4.25) contain the R-independent terms. As far as we know the physical meaning of such terms remains unclear.

V. THERMODYNAMIC ASYMPTOTICS FOR ELECTROMAGNETIC FIELD WITH BOUNDARY CONDITIONS ON AN INFINITE CYLINDER

The calculation of the vacuum energy of electromagnetic field with boundary conditions defined on a cylinder, to say nothing of the temperature corrections, turned out to be more involved problem than the analogous one for a sphere. Therefore the Casimir problem for a cylinder has been considered only in a few papers [10,14,28–31,15]. We again examine three cases: i) perfectly conducting cylindrical shell; ii) solid cylinder with $c_1 = c_2$; iii) dielectric cylinder when $c_1 \neq c_2$. Here we shall use the results of our previous papers [14,15].

A. Perfectly conducting cylindrical shell

In Ref. [14] the first two terms in the uniform asymptotic expansion of the product of the modified Bessel functions $I_n nx\,K_n(nx)$ were taken into account. As a result the spectral zeta function in the problem under consideration was represented as an expansion in terms of the Riemann zeta functions $\zeta_{\rm R}(2(k+s)+1)$ with $k = 0, 1, 2, \ldots$. With allowance for the first two terms in this expansion the zeta function is given by

$$\zeta(s) = Z_1(s) + Z_2(s) + Z_3(s). \tag{5.1}$$

Here the function $Z_1(s)$ stands for the contribution of zero orbital momentum with proper subtraction

$$Z_1(s) = \frac{(2s-1)\,R^{2s-1}}{2\,\sqrt{\pi}\,c^{2s}\,\Gamma(s)\,\Gamma(3/2-s)}\int_0^\infty dy\,y^{-2s}\left\{\ln[1-\mu_0^2(y)] + \frac{1}{4}\,y^2\,t^6(y)\right\}, \tag{5.2}$$

$$\mu_n(y) = y\,\left(I_n(y)\,K_n(y)\right)', \quad t(y) = \frac{1}{\sqrt{1+y^2}}.$$

The function $Z_2(s)$ is generated by the first term of the uniform asymptotic expansion

$$Z_2(s) = \frac{R^{2s-1}}{64\,\sqrt{\pi}\,c^{2s}}\,(1-2s)\,(3-2s)\,[2\,\zeta_{\rm R}(2s+1)+1]\,\frac{\Gamma(1/2+s)}{\Gamma(s)}. \tag{5.3}$$

The function Z_3 corresponds to the the second term of the uniform asymptotic expansion

$$Z_3(s) = \frac{R^{2s-1}}{61440\,\sqrt{\pi}}\,(1-2s)\,(3-2s)\,(784\,s^2 - 104\,s - 235)\,\frac{\Gamma(3/2+s)}{\Gamma(s)}\,\zeta_{\rm R}(2s+3)\,. \tag{5.4}$$

The function $Z_1(s)$ is defined in the strip $-3/2 < \Re\,s < 1/2$. The functions $Z_2(s)$ and $Z_3(s)$ are analytic functions in the whole complex plane s except for the points, where $\Gamma(s)$ and $\zeta_{\rm R}(s)$ have simple poles. In order to find the heat kernel coefficients a_0, $a_{1/2}$, and a_1 trough the relation (3.8) one needs the zeta function defined in the region $1/2+\varepsilon \leq {\rm Re}s \leq 3/2+\varepsilon$ with ε being a positive infinitesimal. However in this region Eq. (5.2) is not applicable directly. In the most simple way we can overcome this difficulty as in the case of perfectly conducting plates by introducing the photon mass μ at the very beginning of the zeta function calculation and making the analytic continuation to the points $s = 1/2,\ 1,\ 3/2$. Upon taking the residua at these points one should put $\mu = 0$.

With regard to all this and using the relation (3.8) we find the heat kernel coefficients

$$a_0 = 0, \quad a_{1/2} = 0, \quad a_1 = 0, \quad a_2 = 0. \tag{5.5}$$

The vanishing heat kernel coefficient a_2 implies that the zeta regularization gives a finite answer for vacuum energy in the problem at hand [14,29]. The coefficient $a_{3/2}$ is determined by the function $Z_2(s)$ only (see Eq. (5.3))

$$\frac{a_{3/2}}{(4\pi)^{3/2}} = \frac{3}{64\,R}. \tag{5.6}$$

The coefficient $a_{5/2}$ is defined by the function $Z_3(s)$ defined in Eq. (5.4)

$$\frac{a_{5/2}}{(4\pi)^{3/2}} = \frac{153}{8192}\,\frac{c^2}{R^3}. \tag{5.7}$$

The calculation of the next heat kernel coefficients $a_3,\ a_{7/2}, \dots$ would demand a knowledge of the additional terms in the expansion of the spectral zeta in the problem under consideration in terms of the Riemann zeta function. These terms are proportional to $\zeta_{\rm R}(2k+2s+1)$ with $k = 2, 3, \dots$, and may be evaluated employing the technique developed in Ref. [14]. Analyzing the pole positions for these Riemann zeta functions it is easy to show that, as well as in the spherical case, we have

$$a_j = 0, \quad j = 3,\ 4\,5\dots.$$

The zeta determinant entering the high temperature asymptotics of free energy (2.10) and entropy (2.14) is calculated in Appendix B

$$\zeta'(0) = \frac{0.45711}{R} + \frac{3}{32\,R}\,\ln\frac{R}{2\,c}. \tag{5.8}$$

Now we are able to construct the high temperature expansions of the thermodynamic functions in the problem under consideration. For the free energy we have

$$F(T) \simeq -0.22856\,\frac{T}{R} - \frac{3\,T}{64\,R}\,\ln\frac{R\,T}{2\,\hbar\,c} - \frac{51}{65536}\,\frac{\hbar^2\,c^2}{R^3\,T} + \mathcal{O}(T^{-3}). \tag{5.9}$$

When comparing Eq. (5.9) with results of other authors one should remember that all the thermodynamic quantities that we obtain in this section are related to a cylinder of unit length. The high temperature asymptotics of the electromagnetic free energy in presence of perfectly conducting cylindrical shell was investigated in Ref. [10]. To make the comparison handy let us rewrite their result as follows

$$F(T) \simeq -0.10362\,\frac{T}{R} - \frac{3\,T}{64\,R}\,\ln\frac{R\,T}{2\,\hbar\,c}. \tag{5.10}$$

The discrepancy between the terms linear in T in Eqs. (5.9) and (5.10) is due to the double scattering approximation used in Ref. [10] (see also the next subsection). Our approach provides an opportunity to calculate the exact value of this term (see Eq. (5.9)).

And finally, making use of the general formulae (2.13) and (2.14) we derive

$$U(T) \simeq \frac{3\,T}{64\,R} - \frac{153}{98304}\,\frac{c^2\,\hbar^2}{R^3\,T} + \mathcal{O}(T^{-3}), \tag{5.11}$$

$$S(T) \simeq \frac{0.27544}{R} + \frac{3}{64\,R}\,\ln\frac{R\,T}{2\,\hbar\,c} - \frac{153}{196608}\,\frac{c^2\,\hbar^2}{R^3\,T^2} + \mathcal{O}(T^{-4}). \tag{5.12}$$

B. Compact cylinder with $c_1 = c_2$ and with $c_1 \neq c_2$

Here we are dealing with two problems: i) cylindrical surface between two media such that the velocity of light is the same inside and outside the cylinder and ii) pure dielectric cylinder with $c_1 \neq c_2$. The explicit expressions for the heat kernel coefficients up to a_2 we take from Ref. [15], where a compact cylinder with unequal velocities of light inside and outside was considered. When $c_1 = c_2$ the final expressions for these coefficients are drastically simplified

$$a_0 = 0, \quad a_{1/2} = 0, \quad a_1 = 0, \quad \frac{a_{3/2}}{(4\pi)^{3/2}} = \frac{3\xi^2}{64\,R}, \quad a_2 = 0. \tag{5.13}$$

The zeta function obtained for given boundary conditions in Ref. [14] gives

$$\frac{a_{5/2}}{(4\pi)^{3/2}} = \xi^2 \frac{c^2}{R^3}\frac{45}{8192}, \quad a_j = 0, \;\; j = 3, 4, 5, \dots . \tag{5.14}$$

The heat kernel coefficients (5.13) and (5.14) lead to the following high temperature behavior of the internal energy in the problem at hand

$$U(T) = \frac{3\xi^2 T}{64\,R}\left(1 - \frac{5}{512}\frac{c^2\hbar^2}{R^2T^2}\right) + \mathcal{O}(T^{-3})\,. \tag{5.15}$$

The corresponding zeta determinant is calculated in Appendix B

$$\zeta'(0) = \frac{\xi^2}{R}\left(0.20483 + \frac{3}{32}\,\ln\frac{R}{2\,c}\right). \tag{5.16}$$

Now we can write the high temperature asymptotics for free energy

$$F(T) = -\xi^2\frac{T}{R}0.10242 - \xi^2\frac{3T}{64R}\ln\frac{TR}{2\hbar c} - \xi^2\frac{15}{65536}\frac{c^2\hbar^2}{R^3T} + \mathcal{O}(T^{-3}) \tag{5.17}$$

and for entropy

$$S(T) = \frac{\xi^2}{R}\left[0.10242 + \frac{3}{64}\left(1 + \ln\frac{RT}{2\hbar c}\right) - \frac{15}{65536}\frac{c^2\hbar^2}{T^2R^2}\right] + \mathcal{O}(T^{-4})\,. \tag{5.18}$$

Putting in these equations $\xi^2 = 1$ we arrive at the double scattering approximation for a perfectly conducting cylindrical shell (see Eq. (5.10)). A slight distinction between the linear in T terms in Eq. (5.10) and Eq. (5.17) is due to an approximate numerical evaluation of the zeta determinant in Eq. (B12).

In the case of a pure dielectric cylinder ($\mu_1 = \mu_2 = 1$, $\varepsilon_1 \neq \varepsilon_2$) the first four heat kernel coefficients are different from zero even in the dilute approximation [15] (small differences between the velocities of light inside and outside the cylinder)

$$\begin{aligned} a_0 = -\frac{6\,\pi\,R^2}{c_2^4}\,(c_1 - c_2) + \frac{12\,\pi\,R^2}{c_2^5}\,(c_1 - c_2)^2, \quad a_{1/2} &= \frac{-2\,\pi^{3/2}\,R}{c_2^4}\,(c_1 - c_2)^2, \\ a_1 = \frac{8\,\pi}{c_2^2}\,(c_1 - c_2) - \frac{14\,\pi}{3\,c_2^3}\,(c_1 - c_2)^2, \quad a_{3/2} &= \frac{3\,\pi^{3/2}}{16R\,c_2^2}\,(c_1 - c_2)^2, \\ a_2 = 0, \quad \frac{a_{5/2}}{(4\pi)^{3/2}} &= \frac{857}{61440}\frac{(c_1 - c_2)^2}{R^3}. \end{aligned} \tag{5.19}$$

It should be noted that the coefficient a_2 vanishes only in the $(c_1 - c_2)^2$-approximation. As a matter of fact a_2 contains nonvanishing $(c_1 - c_2)^3$-terms and those of higher order [15]. Therefore the zeta regularization provides a finite answer for the vacuum energy of a pure dielectric cylinder only in the $(c_1 - c_2)^2$-approximation even at zero temperature.

The contribution to the asymptotic expansions of the first three heat kernel coefficients should be "absorbed" by the renormalization of the relevant phenomenological parameters in the general expression of the classical energy of a dielectric cylinder (in the same way as it has been done for a pure dielectric ball). By making use of the coefficient $a_{3/2}$ we get the high temperature asymptotics of the internal energy in the problem at hand

$$U(T) = \Delta n^2 \frac{3}{128}\frac{T}{R}\left(1 - \frac{857}{17280}\frac{c^2\hbar^2}{T^2R^2}\right) + \mathcal{O}(T^{-2})\,. \tag{5.20}$$

where $\Delta n = n_1 - n_2 \simeq (c_2 - c_1)/c$.

In view of a considerable technical difficulties we shall not calculate the zeta function determinant for a pure dielectric cylinder. We recover the respective asymptotics of free energy by integrating the thermodynamic relation (2.11) and of entropy by using the relation (2.12). Pursuing this way we introduce a new constant of integration α that remaines undetermined

$$F(T) = -\Delta n^2 \frac{3}{128}\frac{T}{R}\left(\alpha + \ln\frac{RT}{\hbar c} + \frac{857}{34560}\frac{c^2\hbar^2}{T^2R^2}\right) + \mathcal{O}(T^{-2})\,, \tag{5.21}$$

$$S(T) = \Delta n^2 \frac{3}{128}\left(1 + \alpha + \ln\frac{RT}{\hbar c} - \frac{857}{34560}\frac{c^2\hbar^2}{T^2R^2}\right) + \mathcal{O}(T^{-2})\,. \tag{5.22}$$

VI. CONCLUSIONS

In this paper we have demonstrated efficiency and universality of the high temperature expansions in terms of the heat kernel coefficients for the Casimir problems with spherical and cylindrical symmetries. All the known results in this field are reproduced in a uniform approach and in addition a few new asymptotics are derived (for a compact ball with $c_1 = c_2$ and for a pure dielectric infinite cylinder).

As the next step in the development of this approach one can try to retain the terms exponentially decreasing when $T \to \infty$. These corrections are well known, for example, for thermodynamic functions of electromagnetic field in the presence of perfectly conducting parallel plates [20,32]. In order to reveal such terms, first of all the exponentially decreasing corrections should be retained in the asymptotic expansion (2.8) for the heat kernel.

As in all the Casimir calculations which take into account the material characteristics of the boundaries, in the approach under consideration the renormalization procedure should be formulated exactly in the framework of the high temperature expansions (2.10), (2.13), and (2.14). The explicit divergencies are not encountered in these asymptotics, and nevertheless the renormalization should be carried out as it was shown in this paper when considering a pure dielectric ball and a dielectric cylinder.

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APPENDIX A: ZETA FUNCTION DETERMINANTS FOR ELECTROMAGNETIC FIELD SUBJECTED TO SPHERICALLY SYMMETRIC BOUNDARY CONDITIONS

1. A perfectly conducting sphere

First we calculate $\zeta'(0)$ (zeta determinant) for electromagnetic field in the background of a perfectly conducting sphere. We proceed from the following representation for this zeta function [14]

$$\zeta(s) = \left(\frac{a}{c}\right)^{2s}\frac{\sin(\pi\, s)}{\pi}\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty} dy\, y^{-2s}\,\frac{d}{d\,y}\,\ln[1-\sigma_l^2(y)]\,, \tag{A1}$$

where

$$\sigma_l(y) = \frac{d}{d\,y}[y\, I_\nu(y)\, K_\nu(y)], \qquad \nu = l + 1/2. \tag{A2}$$

The analytic continuation of the formula (A1) to the region $\Im s < 0$ is performed by adding and subtracting in the integrand (A1) its asymptotics at large ν

$$\sigma_l^2(\nu\, y) \simeq \frac{t^6(y)}{4\nu^2}, \qquad t(y) = \frac{1}{\sqrt{1+y^2}}\,. \tag{A3}$$

As a result we obtain

$$\begin{aligned}\zeta(s) = {} & \left(\frac{R}{c}\right)^{2s} \frac{\sin \pi s}{2\pi} \sum_{l=1}^{\infty} \nu^{1-2s} \int_0^\infty \frac{dy}{y^{2s}} \frac{d}{dy} \left\{ \ln[1-(\sigma'_\nu(\nu y))^2] + \frac{1}{4\,\nu^2}\frac{1}{(1+y^2)^3} \right\} \\ & + \frac{1}{4}\left(\frac{R}{c}\right)^{2s} s(1+s)(2+s)\left[(2^{1+2s}-1)\zeta_{\mathrm{R}}(1+2s) - 2^{1+2s}\right]. \end{aligned} \tag{A4}$$

Differentiation of Eq. (A4) at the point $s=0$ gives

$$\begin{aligned}\zeta'(0) = {} & -2\sum_{l=1}^{\infty} \nu \left\{ \ln[1-(\sigma'_\nu(\nu y))^2]|_{y=0} + \frac{1}{4\,\nu^2} \right\} \\ & + \left(-\frac{5}{8} + \frac{1}{2}\ln\frac{R}{c} + \ln 2 + \frac{\gamma}{2} \right). \end{aligned} \tag{A5}$$

Taking into account that $\lim_{y=0} \sigma'_\nu(\nu y) = 1/(2\,\nu)$ one can rewrite Eq. (A5) as follows

$$\zeta'(0) = -2\sum_{l=1}^{\infty} \nu \left[\ln\left(1 - \frac{1}{4\nu^2}\right) + \frac{1}{4\,\nu^2}\right] + \left(-\frac{5}{8} + \frac{1}{2}\ln R + \ln 2 + \frac{\gamma}{2} \right). \tag{A6}$$

In order to find the sum over l in Eq. (A6) let us consider an auxiliary sum

$$\sum_{l=1}^{\infty} 2\,\nu \left\{ \ln\left[1-\frac{a^2}{4\nu^2}\right] + \frac{a^2}{4\,\nu^2} \right\}, \tag{A7}$$

where a is a parameter. Differentiation of this sum with respect to a gives

$$\begin{aligned}\sum_{l=1}^{\infty} 2\,\nu \frac{d}{d\,a}\left[\ln\left(1-\frac{a^2}{4\nu^2}\right) + \frac{a^2}{4\,\nu^2}\right] & = -a\,(2-\gamma-2\ln 2) \\ & + \frac{a}{2}\psi\left(\frac{a}{2}+\frac{3}{2}\right) + \frac{a}{2}\psi\left(-\frac{a}{2}+\frac{3}{2}\right), \end{aligned} \tag{A8}$$

where $\psi(x)$ is the digamma function (the Euler ψ function): $\psi(x) = (d/dx)\ln\Gamma(x)$. Now we integrate the both sides of Eq. (A8) over a from 0 to 1

$$\begin{aligned}\sum_{l=1}^{\infty} 2\,\nu \left[\ln\left(1-\frac{1}{4\nu^2}\right) + \frac{1}{4\,\nu^2}\right] & = \int_0^1 da\,\frac{a}{2}\left[(2\,\gamma + 4\ln 2 - 4) + \psi\left(\frac{a}{2}+\frac{3}{2}\right) + \psi\left(-\frac{a}{2}+\frac{3}{2}\right)\right] \\ & = -\frac{1}{2} + \frac{\gamma}{2} - \frac{7}{6}\ln 2 - 6\,\zeta'_{\mathrm{R}}(-1). \end{aligned} \tag{A9}$$

Substituting this result into Eq. (A6) we get

$$\begin{aligned}\zeta'(0) & = \frac{1}{2} - \frac{\gamma}{2} + \frac{7}{6}\ln 2 + 6\,\zeta'_{\mathrm{R}}(-1) + \left(-\frac{5}{8} + \frac{1}{2}\ln\frac{R}{c} + \ln 2 + \frac{\gamma}{2}\right) \\ & = 0.384292 + \frac{1}{2}\ln\frac{R}{c}. \end{aligned} \tag{A10}$$

2. A material ball with $c_1 = c_2$

The same technique can be used for calculating the zeta function determinant in the case of equal velocities of light inside and out side the material ball (see Section 3.2). The complete zeta function in this problem has the form [14]

$$\zeta(s) = \left(\frac{R}{c}\right)^{2s} \frac{\sin(\pi\, s)}{\pi} \sum_{l=1}^{\infty} (2l+1) \int_0^\infty dy\, y^{-2s} \frac{d}{d\,y} \ln[1 - \xi^2 \sigma_l^2(y)], \tag{A11}$$

where $\sigma_l(y)$ is defined in Eq. (A 1) and the parameter ξ^2 was introduced in Eq. (4.12). Adding and subtracting under the integral sign in Eq. (A11) the integrand asymptotics at large ν we get

$$\begin{aligned}\zeta(s) = \left(\frac{R}{c}\right)^{2s} \frac{\sin(\pi s)}{2\pi} \sum_{l=1}^{\infty} \nu^{-2s+1} \int_0^\infty \frac{dy}{y^{2s}} \frac{d}{dy} \left\{ \ln[1 - \xi^2 \left(\sigma_\nu'(\nu y)\right)^2] + \frac{\xi^2}{4\,\nu^2} \frac{1}{(1+y^2)^3} \right\} \\ - \frac{3\, R^{2s}\, \xi^2}{2\,\Gamma(s+1)\,\Gamma(-s)} \left[\zeta(2s+1, 1/2) - 2^{2s+1}\right] \mathrm{B}(1-s, s+3).\end{aligned} \tag{A12}$$

The derivative of the zeta function at the point $s = 0$, is the following

$$\zeta_{ball}'(s)|_{s=0} = -2 \sum_{l=1}^{\infty} \nu \left\{ \ln\left[1 - \frac{\xi^2}{4\nu^2}\right] + \frac{\xi^2}{4\,\nu^2} \right\} + \xi^2 \left\{ -\frac{5}{8} + \frac{1}{2} \ln\frac{R}{c} + \ln 2 + \frac{\gamma}{2} \right\}. \tag{A13}$$

Restricting ourselves with the first order of ξ^2 we arrive at final result

$$\zeta_{ball}'(s)|_{s=0} = \xi^2 \left\{ -\frac{5}{8} + \frac{1}{2} \ln\frac{R}{c} + \ln 2 + \frac{\gamma}{2} \right\}. \tag{A14}$$

3. A pure dielectric ball

The case of material ball with arbitrary speeds of light inside and outside discussed in Section 3.3 is more complicated. In notations of the paper [26] the zeta function takes the form $\zeta(s) = \zeta_{-1}(s) + \zeta_1(s)$, where

$$\zeta_\rho(s) = -\frac{2\, R^{2s}}{\Gamma(s+1)\,\Gamma(-s)} \sum_{l=1}^{\infty} \nu^{-2s+1} \int_0^\infty dk\, k^{-2s} \frac{\partial}{\partial k} \ln \Delta_{\rho,l}(\nu k), \quad \nu = l + \frac{1}{2}, \quad \rho = \pm 1, \tag{A15}$$

with

$$\Delta_{\rho,l}(\nu k) = \frac{2\, e^{-(k_1 - k_2)\,\nu}}{(\chi^\rho + 1)} \left[\chi^\rho s_l'(\nu k_1)\, e_l(\nu k_2) - s_l(\nu k_1)\, e_l'(\nu k_2)\right], \quad k_1 = k/c_1, \;\; k_2 = k/c_2, \tag{A16}$$

$$s_l(y) = \sqrt{\frac{\pi y}{2}}\, I_\nu(y), \quad e_l(y) = \sqrt{\frac{2\,y}{\pi}}\, K_\nu(y).$$

The parameter $\chi = \sqrt{(\varepsilon_1\, \mu_2)/(\varepsilon_2\, \mu_1)}$ corresponds to ξ in [26].

The analytic continuation of the zeta function to the region $s > 0$ is performed by adding and subtracting from (A16) several terms of its asymptotic expansion

$$\Delta_{\rho,l}(\nu k) \sim \sum_{n=-1,0,1}^{\infty} \frac{D_{n,\rho}}{\nu^n}, \tag{A17}$$

where

$$\begin{aligned} D_{-1} &= \eta(k_1) - \eta(k_2) - (k_1 - k_2), \quad \eta(z) = \sqrt{1+z^2} + \ln\frac{z}{1+\sqrt{1+z^2}}, \\ D_0 &= \ln\left\{ \frac{\chi^\rho\, c_1\, t_2 + c_2\, t_1}{\sqrt{c_1\, c_2\, t_1\, t_2}\, (\chi^\rho + 1)} \right\}, \quad t_i = \frac{1}{\sqrt{1+k_i^2}}, \quad i = 1,\ 2. \end{aligned} \tag{A18}$$

For our purpose it is sufficient to consider four leading terms of the asymptotic expansion (A17), $n = -1, 0, 1, 2$. The functions D_1 and D_2 are presented in [26].

The zeta function (A15) may be rewritten as follows

$$\zeta_\rho(s) = -\frac{2\,R^{2s}}{\Gamma(s+1)\,\Gamma(-s)} \sum_{l=1}^{\infty} \nu^{-2s+1} \int_0^\infty dk\, k^{-2s} \frac{\partial}{\partial k} \left\{ \ln \Delta_{\rho,l} - \nu\, D_{-1} - D_0 - \frac{D_1}{\nu} - \frac{D_2}{\nu^2} \right\}$$

$$-\frac{2\,R^{2s}}{\Gamma(s+1)\,\Gamma(-s)} \left\{ \zeta_H\left(2s-2, \frac{3}{2}\right) \int_0^\infty dk\, k^{-2s} \frac{\partial D_{-1}}{\partial k} + \zeta_H\left(2s-1, \frac{3}{2}\right) \int_0^\infty dk\, k^{-2s} \frac{\partial D_0}{\partial k} \right.$$

$$\left. +\zeta_H\left(2s, \frac{3}{2}\right) \int_0^\infty dk\, k^{-2s} \frac{\partial D_1}{\partial k} + \zeta_H\left(2s+1, \frac{3}{2}\right) \int_0^\infty dk\, k^{-2s} \frac{\partial D_2}{\partial k} \right\}, \tag{A19}$$

Taking the derivative of the zeta function (A19) at the point $s = 0$ with account for the behavior of $D_i(k)$ at $k = 0$ and $k = \infty$ we obtain

$$\zeta_\rho'(0) = -2 \sum_{l=1}^{\infty} \left(l + \frac{1}{2}\right) \left\{ \ln\left[1 + \frac{1}{2\nu} \frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2}\right] - \frac{1}{2\nu} \frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2} + \frac{1}{8\nu^2} \left(\frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2}\right)^2 \right\}$$

$$+2 \left\{ \frac{1}{4} \ln\frac{c_2}{c_1} + \frac{11}{24} \ln\left[\frac{\chi^\rho c_1 + c_2}{\sqrt{c_1 c_2}\,(\chi^\rho + 1)}\right] + \frac{1}{2} \frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2} \right.$$

$$\left. -\frac{1}{8}(2 - \ln R - \gamma - 2\ln 2) \left(\frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2}\right)^2 - \int_0^\infty dk\, \ln k \frac{\partial}{\partial k} D_2 \right\} \tag{A20}$$

In order to derive the sum over l in (A20) let us consider the auxiliary sum

$$\sum_{l=1}^{\infty} 2\,\nu \left\{ \ln\left[1 + \frac{b}{2\,\nu}\right] - \frac{b}{2\,\nu} + \frac{b^2}{8\,\nu^2} \right\}. \tag{A21}$$

Taking the derivative over the parameter b one obtains

$$\sum_{l=1}^{\infty} 2\,\nu \frac{d}{d\,b} \left\{ \ln\left[1 + \frac{b}{2\,\nu}\right] - \frac{b}{2\,\nu} + \frac{b^2}{8\,\nu^2} \right\} = -\frac{b}{2}(2 - \gamma - 2\ln 2) + \frac{b}{2}\Psi\left(\frac{a}{2} + \frac{3}{2}\right). \tag{A22}$$

The integration of the left-hand side in (A22) over b from 0 to $b = (\chi^\rho c_1 - c_2)/(\chi^\rho c_1 + c_2)$ gives the sum entering (A20). Thus

$$\sum_{l=1}^{\infty} 2\,\nu \left\{ \ln\left[1 + \frac{b}{2\,\nu}\right] - \frac{b}{2\,\nu} + \frac{b^2}{8\,\nu^2} \right\} = \frac{b^2}{4}(-2 + \gamma + 2\ln 2) + b\,\zeta_R'\left(0, \frac{3}{2} + \frac{b}{2}\right)$$

$$-2\left[\zeta_R\left(-1, \frac{3}{2} + \frac{b}{2}\right) + \zeta_R'\left(-1, \frac{3}{2} + \frac{b}{2}\right)\right] + 2\left[\zeta_R\left(-1, \frac{3}{2}\right) + \zeta_R'\left(-1, \frac{3}{2}\right)\right] \tag{A23}$$

Inserting (A23) into (A20) we arrive at

$$\zeta_\rho'(0) = -\frac{b^2}{4}(-2 + \gamma + 2\ln 2) - b\,\zeta_R'\left(0, \frac{3}{2} + \frac{b}{2}\right)$$

$$+2\left[\zeta_R\left(-1, \frac{3}{2} + \frac{b}{2}\right) + \zeta_R'\left(-1, \frac{3}{2} + \frac{b}{2}\right)\right] - 2\left[\zeta_R\left(-1, \frac{3}{2}\right) + \zeta_R'\left(-1, \frac{3}{2}\right)\right]$$

$$+2\left\{ \frac{1}{4}\ln\frac{c_2}{c_1} + \frac{11}{24}\ln\left[\frac{\chi^\rho c_1 + c_2}{\sqrt{c_1 c_2}\,(\chi^\rho + 1)}\right] + \frac{b}{2} - \frac{b^2}{8}(2 - \ln R - \gamma - 2\ln 2) \right.$$

$$\left. - \int_0^\infty dk\, \ln k \frac{\partial}{\partial k} D_2 \right\}, \qquad b = \frac{\chi^\rho c_1 - c_2}{\chi^\rho c_1 + c_2}. \tag{A24}$$

In the case of nonmagnetic media ($\mu_1 = \mu_1 = 1$) the right-hand side of (A24) is slightly simplified, and one can expand the zeta function derivative (functional determinant) with respect to the difference $(c_1 - c_2)$, where $c_1 = 1/\sqrt{\varepsilon_1}$, $c_2 = 1/\sqrt{\varepsilon_2}$. Consequently we get

$$\zeta'(0) = \zeta_{\rho=-1}'(0) + \zeta_{\rho=1}'(0) = \frac{1}{4\,c_2^2}\left[-\frac{7}{8} + \ln\frac{R}{c_2} + \ln 4 + \gamma\right](c_1 - c_2)^2 + \mathcal{O}((c_1 - c_2)^3). \tag{A25}$$

APPENDIX B: ZETA FUNCTION DETERMINANTS FOR ELECTROMAGNETIC FIELD WITH CYLINDRICALLY SYMMETRIC BOUNDARY CONDITIONS

1. A perfectly conducting cylindrical shell

This Appendix is devoted to the calculation of the zeta function derivatives for the electromagnetic field in presence of a perfectly conducting cylindrical shell and of a material cylinder, such that the speed of light inside it is equal to the speed of light outside it. We start from the first case. The zeta function looks like [14]

$$\zeta(s) = \frac{c^{-2s}\,R^{2s-1}}{\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)}\int_0^\infty dy\,y^{1-2*s}\,\frac{d}{d\,y}\,\ln[1-\varrho_0^2(y)]$$
$$+\frac{c^{-2s}\,R^{2s-1}}{\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)}\sum_{n=1}^\infty n^{1-2s}\int_0^\infty dy\,y^{1-2s}\,\frac{d}{d\,y}\,\ln[1-\varrho_n^2(ny)], \tag{B1}$$

where

$$\varrho_l(y) = y\,\frac{d}{d\,y}[I_n(y)\,K_n(y)].$$

The analytic continuation to the region $s<0$ is performed by adding and subtracting from the integrand (B1) the first term of its asymptotics at large n

$$\ln[1-\varrho_l^2(\nu\,y)] \simeq -\frac{y^4\,t^6(y)}{4\,n^2} - \frac{y^4\,t^8}{16\,n^2}\left(3-30\,t^2+35\,t^4+\frac{1}{2}\,y^4\,t^4\right)+\mathcal{O}(n^{-6}). \tag{B2}$$

As a result we obtain

$$\zeta(s) = \frac{c^{-2s}\,R^{2s}}{2\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)}\int_0^\infty \frac{dy}{y^{2s-1}}\,\frac{\partial}{\partial y}\,\ln[1-\varrho_0^2(y)]$$
$$+\frac{c^{-2s}\,R^{2s}}{\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)}\sum_{n=1}^\infty n^{1-2s}\int_0^\infty \frac{dy}{y^{2s-1}}\,\frac{\partial}{\partial y}\left[\ln(1-\varrho_n^2(n\,y))+\frac{y^4\,t^6}{4\,n^2}\right]$$
$$-\frac{c^{-2s}\,R^{2s}}{\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)}\,\zeta_R(2s+1)\,\frac{2s-1}{16}\,\Gamma(s+1/2)\,\Gamma(5/2-s) \tag{B3}$$

The derivative of the zeta function (B3) at the point $s=0$, is of the form

$$\left.\frac{d}{ds}\zeta(s)\right|_{s=0} = \frac{1}{\pi\,R}\int_0^\infty dy\,y\,\frac{d}{dy}\,\ln[1-\varrho_0^2(y)] \tag{B4}$$
$$+\frac{2}{\pi\,R}\sum_{n=1}^\infty n\int_0^\infty dy\,y\,\frac{d}{dy}\left[\ln(1-\varrho_n^2(n\,y))+\frac{y^4\,t^6}{4\,n^2}\right]-\frac{1}{32\,R}\left(-3\gamma+3\,\ln\frac{2\,c}{R}+4\right).$$

Unfortunately, the differentiation with respect to s at the point $s=0$ does not remove the y-integration, as it did in the case of spherically symmetric boundaries. Therefore, the first term in (B4) is calculated numerically. To estimate the second one, we apply the asymptotics (B2)

$$\frac{2}{\pi\,R}\sum_{n=1}^\infty n\int_0^\infty dy\,y\,\frac{d}{dy}\left[\ln(1-\varrho_n^2(y))+\frac{y^4\,t^6}{4\,n^2}\right]\simeq$$
$$\simeq -\frac{1}{8\,\pi\,R}\sum_{n=1}^\infty\frac{1}{n^3}\int_0^\infty dy\,y\,\frac{d}{dy}\left[y^4\,t^8\left(3-30\,t^2+35\,t^4+\frac{1}{2}\,t^4\,y^4\right)\right]$$
$$= -\frac{\zeta(3)}{8\,R}\,\frac{47}{1024} = -\frac{0.0069}{R} \tag{B5}$$

Substituting this into (B4) one obtains

$$\zeta'(0) = \frac{0.45711}{R}+\frac{3}{32\,R}\,\ln\frac{R}{2\,c}. \tag{B6}$$

2. A compact infinite cylinder with $c_1 = c_2$

Now we turn to the case of material cylinder placed into unbounded medium, such that the speed of light inside the cylinder is equal to the speed of light outside it. Proceeding as in the case of cylindrical shell we start from the expression

$$\zeta_{cyl}(s) = \frac{c^{-2s}\, R^{2s-1}}{2\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)} \sum_{n=-\infty}^{\infty} \int_0^\infty dy\, y^{1-2s}\, \frac{d}{d\,y} \ln[1-\xi^2 \varrho_n^2(y)], \tag{B7}$$

within the first order of ξ^2-expansion assuming the form

$$\begin{aligned}\zeta_{cyl}(s) =& -\frac{c^{-2s}\, R^{2s-1}\,\xi^2}{2\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)} \int_0^\infty dy\, y^{1-2s}\, \frac{d}{d\,y}\, \varrho_0^2(y) \\ & -\frac{c^{-2s}\, R^{2s-1}\,\xi^2}{2\sqrt{\pi}\,\Gamma(s)\,\Gamma(3/2-s)} \sum_{n=1}^{\infty} \int_0^\infty dy\, y^{1-2s}\, \frac{d}{d\,y}\, \varrho_n^2(n\,y)]\end{aligned} \tag{B8}$$

The analytic continuation to $s<0$ is obtained in the same way as for the cylindrical shell by adding and subtracting from the integrand (B7) the first term of its asymptotics at large n.

$$-\varrho_l^2(\nu\, y) \simeq -\frac{y^4\, t^6(y)}{4\, n^2} - \frac{y^4\, t^8}{16\, n^2}\left(3 - 30\, t^2 + 35\, t^4\right) + \mathcal{O}(n^{-6}). \tag{B9}$$

Taking the derivative of the analytic continuation at $s=0$ we arrive at

$$\begin{aligned}\left.\frac{d}{ds}\zeta_{cyl}(s)\right|_{s=0} =& -\frac{\xi^2}{\pi\, R}\int_0^\infty dy\, y\, \frac{d}{dy}\, \varrho_0^2(y) + \frac{2\,\xi^2}{\pi\, R}\sum_{n=1}^{\infty} n \int_0^\infty dy\, y\, \frac{d}{dy}\left[-\varrho_n^2(n\,y)) + \frac{y^4\, t^6}{4\, n^2}\right] \\ & -\frac{\xi^2}{32\, R}\left(-3\gamma + 3\,\ln\frac{2\,c}{R} + 4\right)\end{aligned} \tag{B10}$$

The first term in Eq. (B10) is calculated numerically. Applying the asymptotics (B9), we estimate the second one

$$\begin{aligned}& \frac{2\,\xi^2}{\pi\, R}\sum_{n=1}^{\infty} n \int_0^\infty dy\, y\, \frac{d}{dy}\left[-\varrho_n^2(n\,y) + \frac{y^4\, t^6}{4\, n^2}\right] \simeq \\ & \simeq -\frac{\xi^2}{8\,\pi\, R}\sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^\infty dy\, y\, \frac{d}{dy}\left[y^4\, t^8\, \left(3 - 30\, t^2 + 35\, t^4\right)\right] \\ & = -\xi^2 \frac{\zeta(3)}{8\, R}\, \frac{27}{512} = -0.0079\, \frac{\xi^2}{R}\end{aligned} \tag{B11}$$

Consequently,

$$\begin{aligned}\zeta'(0) &= \frac{\xi^2}{R}\left[0.28364 - 0.00792 - \frac{1}{32}\left(-3\gamma + 4 + 3\,\ln\frac{2\,c}{R}\right)\right] \\ &= \frac{\xi^2}{R}\left(0.20483 + \frac{3}{32}\,\ln\frac{R}{2\,c}\right).\end{aligned} \tag{B12}$$

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